Quantum expanders and growth of group representations

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Abstract

Let π be a finite dimensional unitary representation of a group G with a generating symmetric n-element set $S \subset G$. Fix $\varepsilon > 0$. Assume that the spectrum of $|S|^{-1} \sum_{s \in S} \pi(s) \otimes \overline{\pi(s)}$ is included in $[-1, 1-\varepsilon]$ (so there is a spectral gap $\geq \varepsilon$). Let $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leq N$ that appear in π. Then let $R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$ where the supremum runs over all π with n, ε fixed. We prove that there are positive constants δ_ε and c_ε such that, for all sufficiently large integer n (i.e. $n \geq n_0$ with n_0 depending on ε) and for all $N \geq 1$, we have $\exp \delta_\varepsilon n N^2 \leq R'_{n,\varepsilon}(N) \leq \exp c_\varepsilon n N^2$. The same bounds hold if, in $r'_N(\pi)$, we count only the number of distinct irreducible representations of dimension exactly $= N$.

1 Introduction

We wish to formulate and answer a natural extension of a question raised explicitly by Wigderson in several lectures (see e.g. [23, p.59]) and also implicitly in [18]. Although the variant that we answer seems to be much easier, it may shed some light on the original question. Wigderson's question concerns the growth of the number $r_N(G)$ of distinct irreducible representations of dimension $\leq N$ that may appear on a finite group G when the order of G is arbitrarily large and all that one knows is that G admits a generating set S of n elements for which the Cayley graph forms an expander with a fixed spectral gap $\varepsilon > 0$. The problem is to find the best bound of the form $r_N(G) \leq R(N)$ with $R(N)$ independent of the order of G (but depending on n, ε). We consider a more general framework: the finite group G is replaced by a finite dimensional representation π (playing the role of the regular representation λ_G for finite groups) such that the representation $\pi \otimes \bar{\pi}$ admits a spectral gap, meaning that the trivial representation is isolated with a gap $\geq \varepsilon$ from the other irreducible components of $\pi \otimes \bar{\pi}$. When $\pi = \lambda_G$ we recover the previous notion of spectral gap. Let then $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leq N$ appearing in π (note that $r_N(G) = r'_N(\lambda_G)$), and let $R'(N)$ denote the least upper bound $r'_N(\pi) \leq R'(N)$ when the only restriction on π is that n, ε remain fixed (but the dimension of π is arbitrary). We observe that the previously known bound for $R(N)$ namely $R(N) = e^{O(nN^2)}$ is also valid for $R'(N)$

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and also that $R(N) \le R'(N)$. Our main result, which follows from the metric entropy estimate for quantum expanders in [20], is that this bound for $R'(N)$ is sharp: there is $\delta > 0$ such that for all n large enough (i.e. $\forall n \ge n_0(\varepsilon)$) we have $R'(N) \ge e^{\delta n N^2}$ for all N.

The term "quantum expander" was coined in [9, 2, 3] to which we refer for background (see also [7, 8]).

2 Main result

Let G be any group with a finite generating set $S \subset G$ with $|S| = n$. For any unitary representation $\pi:\ G \to H_\pi$ we set

$$\lambda(\pi, S) = n^{-1} \sup\{\Re\ \langle \sum_{s\in S} \pi(s)\xi, \xi\rangle \mid \xi \in H_\pi^{\mathrm{inv}\perp}, \|\xi\|_{H_\pi} = 1\}.$$

where $H_\pi^{\mathrm{inv}} \subset H_\pi$ denotes the subspace of all π-invariant vectors.
When S is symmetric, $\sum_{s\in S} \pi(s)$ being selfadjoint, the real part sign $\Re$ can be omitted.
We then set

$$\varepsilon(\pi, S) = 1 - \lambda(\pi, S).$$

It will be useful to record here the elementary observation that if π is unitarily equivalent to the direct sum $\oplus_{i\in I}\pi_i$ of a family of unitary representations, then $\lambda(\pi, S) = \sup_{i\in I} \lambda(\pi_i, S)$ and hence

$$\varepsilon(\pi, S) = \inf_{i\in I} \varepsilon(\pi_i, S). \tag{2.1}$$

In particular, if π_1 is contained in π_2, then $\varepsilon(\pi_1, S) \ge \varepsilon(\pi_2, S)$.

We denote

$$\varepsilon(G, S) = \inf\{\varepsilon(\pi, S)\}$$

where the infimum runs over all unitary representations $\pi:\ G \to H_\pi$. Thus the condition

$$\varepsilon(G, S) > 0$$

means that G has Kazhdan's "property (T)", (or in otherwords is a Kazhdan-group), see [1] for more background.

We start by the following result somewhat implicitly due to S. Wassermann [22] and explicitly proved in detail in [6].

Proposition 2.1 ([22, 6])**.** *For any $\varepsilon > 0$ there is a constant c_ε such that for any n, any group G and any $S \subset G$ with $|S| = n$ such that $\varepsilon(G, S) \ge \varepsilon$, the number $r_N(G)$ of distinct irreducible unitary representations $\sigma:\ G \to B(H_\sigma)$ with* $\dim(H_\sigma) \le N$ *is majorized as follows:*

$$r_N(G) \le \exp(c_\varepsilon n N^2). \tag{2.2}$$

Of course, here distinct means up to unitary equivalence.

Remark 2.2. Note that it suffices to prove a bound of the same form for the number of distinct irreducible unitary representations $\sigma:\ G \to B(H_\sigma)$ with $\dim(H_\sigma) = N$. Indeed, if the latter number is denoted by $s_N(G)$, we have $r_N(G) = \sum_{d=1}^N s_d(G)$, so that it suffices to have a bound of the form $s_d(G) \le \exp(c'_\varepsilon n d^2)$ to obtain (2.2).
See [14, 15] for some examples of estimates of the growth of $r_N(G)$.

We note that it was originally proved by Wang [21] that for any Kazhdan-group G this number $r_N(G)$ is finite for any N. There is an indication of proof of (2.2) in [22], and detailed proofs appear in [6] (see also [18]). We will prove a simple extension of this bound below.

Recall that a sequence (G_k, S_k) of finite groups equipped with generating sets $S_k \subset G_k$ such that

$$\sup_k |S_k| < \infty, \quad |G_k| \to \infty \quad \text{and} \quad \inf_k \varepsilon(G_k, S_k) > 0$$

is called an expander or an expanding family. This corresponds to the usual notion among *Cayley* graphs to which we restrict the entire discussion.
Let $\hat{G}$ denote as usual the (finite) set of all irreducible unitary representations of a finite group G (up to unitary equivalence).
We note in passing that it is well known (and this also can be derived from Proposition 2.1) that any expander satisfies

$$\lim_{k\to\infty} \max\{\dim(H_\sigma) \mid \sigma \in \hat{G_k}\} = \infty. \tag{2.3}$$

We refer the reader to the surveys [10, 17] for more information on expanders.

The question raised by Wigderson in this context can be formulated as follows:

Let

$$R_{n,\varepsilon}(N) = \sup\{r_N(G)\}$$

where the supremum runs over all finite groups G admitting a subset S with $|S| = n$ such that $\varepsilon(G, S) \geq \varepsilon$. Actually the question is just as interesting for arbitrary (Kazhdan) groups G, but it is more natural to restrict to finite groups, because there are infinite Kazhdan groups without *any* (nontrivial) finite dimensional representations.

Moreover, since, for a finite group G, all representations are weakly contained in the left regular representation λ_G, we have clearly by (2.1)

$$\varepsilon(G, S) = \varepsilon(\lambda_G, S). \tag{2.4}$$

By (2.2), we have

$$R_{n,\varepsilon}(N) \leq \exp(c_\varepsilon n N^2). \tag{2.5}$$

and a fortiori simply $R_{n,\varepsilon}(N) = \exp O(N^2)$.
Wigderson asked whether this upper bound can be improved. More explicitly, what is the precise order of growth of $\log R_{n,\varepsilon}(N)$ when $N \to \infty$. Does it grow like N rather than like N^2 ?
The motivation for this question can be summarized like this: In [18, Th. 1.4] an exponential bound $\exp O(N)$ is proved for a special class of groups G (namely monomial groups), admitting a fixed spectral gap with generating sets of very slowly growing size (but not bounded) and it is asked whether the same exponential bound holds in general for $R_{n,\varepsilon}(N)$. Moreover, in a remark following the proof of [18, Th. 1.4], Meshulam and Wigderson observe that for any prime number $p > 2$, there is a group G_p with a generating set of (unbounded) size $\log p$ admitting a fixed spectral gap and such that $r_p(G) \approx 2^p/p$.

Remark 2.3. By classical results, originating in the works of Kazhdan and Margulis (see e.g. [16] or [17, Cor. 2.4]), for any fixed $m \geq 3$, the family $\{SL_m(\mathbb{Z}_p) \mid p \text{ prime}\}$ is an expander, so that we have (for suitable ℓ, δ)

$$R_{\ell,\delta}(N) \geq \sup_p r_N(SL_m(\mathbb{Z}_p)).$$

Similarly, let $\mathcal{G}_k$ denote the symmetric group of all permutations of a k element set. Kassabov [11] proved that the family $\{\mathcal{G}_k \mid k \geq 1\}$ forms an expanding family with respect to subsets $S_k \subset \mathcal{G}_k$ of a fixed size ℓ and a fixed spectral gap $\delta > 0$. Thus we find a lower bound

$$R_{\ell,\delta}(N) \geq \sup_k r_N(\mathcal{G}_k).$$

Quite remarkably, it is proved in [13] that the family itself of *all* non-commutative finite simple groups forms an expander (for some suitable n, ε).

Remark 2.4. However, it seems the resulting lower bounds are still far from being exponential in N. Actually, in many important cases (see *e.g.* [4]), the proof that certain finite groups G give rise to expanders uses the fact that the smallest dimension of a (non-trivial) irreducible representation on G is $\geq c|G|^a$ for some $a > 0$. Then since $|G| = \sum_{\pi \in \hat{G}} \dim(\pi)^2$ the cardinal of $\hat{G}$ is bounded above by $|G|^{1-2a}/c^2$. Therefore, for any $N \geq c|G|^a$ we have $r_N(G) \leq |G|^{1-2a}/c^2 \leq c' N^{(1/a)-2}$, so that the resulting growth implied for $R_{n,\varepsilon}(N)$ is at most polynomial in N. (I am grateful to N. Ozawa for drawing my attention to this point).

Nevertheless, we have:

Remark 2.5. (Communicated by Martin Kassabov). For suitable n, ε the numbers $R_{n,\varepsilon}(N)$ grow faster than any power of N. In fact, we will prove the

Claim : There is an expanding family of Cayley graphs (G_k) of groups generated by 3 elements with a positive spectral gap ε and such that for $N_k = 2^{3k} - 2$, G_k admits 2^{k^2} distinct irreducible representations of dimension N_k.

From this claim follows that $R_{3,\varepsilon}(N_k) \geq 2^{k^2} \geq 2^{(\log(N_k))^2}$, say for all k large enough, and hence

$$R_{n,\varepsilon}(N) \geq 2^{(\log(N))^2} \text{ for infinitely many } N\text{'s}.$$

To prove the claim we use the ideas from [12]. Let $\mathcal{R}_k$ denote the (finite) ring $M_k(F_2)$ of $k \times k$ matrices with entries in the field with 2 elements.

It is known that the cartesian product $\Pi_k = \mathcal{R}_k^{2^{k^2}}$ of $|\mathcal{R}_k| = 2^{k^2}$ copies of $\mathcal{R}_k$ is generated by 3 elements. Indeed, $\mathcal{R}_k$ itself is generated as a ring by two elements, e.g. $a = e_{12}$ and the shift $b = e_{12} + e_{23} + \cdots + e_{k-1k} + e_{k1}$, then Π_k is generated as a ring by $\{A, B, C\}$ where A (resp. B) is the element with all coordinates equal to a (resp. b), and C is such that its coordinates are in one to one correspondence with the elements of $\mathcal{R}_k$. To check this, let $R \subset \Pi_k$ be the ring generated by $\{A, B, C\}$. Note, by the choice of C, the following easy observation: for any coordinate i, there is $x \in R$ such that $x_i = 0$ but $x_j \neq 0$ for all $j \neq i$. For any subset I of the index set let $p_I : R \to \mathcal{R}_k^I$ be the coordinate projection. One can then prove by induction on $m = |I|$ that $p_I(R) = \mathcal{R}_k^I$ for all I. Indeed, assume the fact established for $m-1$. For any I with $|I| = m$ we pick $i \in I$ and we consider the set $\mathcal{I} = \{y \in \mathcal{R}_k^{I\setminus i} \mid (0, y) \in p_I(R)\}$. By the induction hypothesis, $\mathcal{I}$ is an ideal in $\mathcal{R}_k^{I\setminus i}$, but, since $\mathcal{R}_k$ is simple, the above observation implies that $\mathcal{I} = \mathcal{R}_k^{I\setminus i}$, and since a, b generate $\mathcal{R}_k$ we have $p_{\{i\}}(R) = \mathcal{R}_k$, so we obtain $p_I(R) = \mathcal{R}_k^I$.

This implies that the free associative ring $\mathbb{Z}\langle x, y, z\rangle$ (in 3 non-commutative variables) can be mapped onto the product Π_k. Consider now the group $EL_3(\mathbb{Z}\langle x, y, z\rangle)$ generated by the elementary matrices in $GL_3(\mathbb{Z}\langle x, y, z\rangle)$. This is a noncommutative universal lattice in the terminology of [12, 5]. First observe that $EL_3(\mathbb{Z}\langle x, y, z\rangle)$ is generated by 3 elements. Indeed, let α, β generate $SL_3(\mathbb{Z})$. Then α, β, γ will generate $EL_3(\mathbb{Z}\langle x, y, z\rangle)$ where $\gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$. Moreover, by [5, Th.1.1] $EL_3(\mathbb{Z}\langle x, y, z\rangle)$ has Kazhdan's property T. It follows that the groups

$$G_k = EL_3(\Pi_k)$$

have expanding generating sets with 3 elements. But it turns out that G_k can be identified with the product

$$SL_{3k}(F_2)^{2^{k^2}}.$$

Indeed, firstly one easily checks the natural isomorphism $EL_3(\mathcal{R}_k^{2^{k^2}}) \simeq EL_3(\mathcal{R}_k)^{2^{k^2}}$, secondly it is well known that, since F_2 is a field, $EL_n(F_2) = SL_n(F_2)$ for any n, and hence (taking $n = 3k$) we have a natural isomorphism $EL_3(\mathcal{R}_k) = SL_{3k}(F_2)$; this yields the identification $G_k = SL_{3k}(F_2)^{2^{k^2}}$.

To conclude, we will use the fact that $SL_{3k}(F_2)$ admits a nontrivial irreducible representation π with dimension $N_k = 2^{3k} - 2$. (Just consider its action by permutation on the projective space, which has $2^{3k} - 1$ elements; the action is transitive and doubly transitive, therefore the associated Koopman representation π is irreducible and of dimension $2^{3k} - 2$). This immediately produces 2^{k^2} distinct irreducible representations of dimension N_k on $SL_{3k}(F_2)^{2^{k^2}}$. Indeed, it is an elementary fact that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ is a product group, and if $\pi_1, \cdots \pi_m$ are arbitrary nontrivial irreducible representations on the factor groups $\Gamma_1, \cdots, \Gamma_m$, then the representations $\widetilde{\pi}_j$ defined on Γ by $\widetilde{\pi}_j(g) = \pi_j(g_j)$ are distinct (meaning not unitarily equivalent), irreducible on Γ and $\dim(\widetilde{\pi}_j) = \dim(\pi_j)$ for any j. So taking all Γ_j's equal to $SL_{3k}(F_2)$, with $\pi_j = \pi$ and $m = 2^{k^2}$, we obtain the announced claim.

In any case, the problem of finding the correct behaviour of $\log R_{n,\varepsilon}(N)$ (or of $R_{n,\varepsilon}(N)$ itself) when $N \to \infty$ appears to be still wide open.

In this paper we consider a modified version of this question involving "quantum expanders" and show that for this (much easier) modified version, N^2 is the correct order of growth.

The term "quantum expander" was introduced in [9] and [2, 3], independently, to designate a sort of non-commutative, or matricial, analogue of expanders, as follows.

Fix an integer n. Consider an n tuple of $N \times N$ unitary matrices, say $u = (u_j) \in U(N)^n$. We view each of them u_j as a linear operator on the N-dimensional Hilbert space H. Then $u_j \otimes \overline{u_j}$ is naturally viewed as a linear operator on the (Hilbert space sense) tensor product $H \otimes \bar{H}$. Using the (canonical) identification $H^* \simeq \bar{H}$, the tensor product $H \otimes \bar{H}$ can be isometrically identified with the space of linear operators from H to H equipped with the Hilbert-Schmidt norm denoted by $\| \ \|_2$ (sometimes called the Frobenius norm in the present finite dimensional context). Then, the identity operator $Id_H : \ H \to H$ defines a distinguished element of $H \otimes \bar{H}$ that we denote by I.

We set

$$\lambda(u) = n^{-1} \sup\{\Re\langle(\sum_1^n u_j \otimes \bar{u}_j)\xi, \xi\rangle \mid \xi \in H \otimes \bar{H}, \ \xi \perp I, \ \|\xi\|_{H\otimes\bar{H}} = 1\},$$

and

$$\varepsilon(u) = 1 - \lambda(u).$$

In other words, with the preceding identifications, the condition $\varepsilon(u) \geq \varepsilon$ means that for any $x \in M_N$ with $\mathrm{tr}(x) = 0$ we have

$$\Re \sum \mathrm{tr}(u_j x u_j^* x^*) \leq (1 - \varepsilon)\|x\|_2,$$

where $\|x\|_2 = (\mathrm{tr}(x^*x))^{1/2}$.
When $T = \sum_1^n u_j \otimes \bar{u}_j$ is self adjoint (in particular when the set $\{u_1, \cdots, u_n\}$ is selfadjoint) the real part $\Re$ can be omitted in the two preceding lines.

In group theoretic language, if $\pi : \ \mathbf{F}_n \to U(N)$ is the group representation on the free group $\mathbf{F}_n$, equipped with a set of n free generators $S = \{g_1, \cdots, g_n\}$, such that $\pi(g_j) = u_j$ $(1 \leq j \leq n)$, then we have

$$\varepsilon(u) = \varepsilon(\pi \otimes \overline{\pi}, S).$$

Definition 2.6. A sequence $\{u(k) \mid k \in \mathbb{N}\}$ with each $u(k) \in U(N_k)^n$ such that $N_k \to \infty$ (with n remaining fixed) and $\inf_k\{\varepsilon(u(k))\} > 0$ is called a quantum expander. We say that n is its degree and $\inf_k\{\varepsilon(u(k))\} > 0$ its spectral gap.

Remark 2.7. The existence of quantum expanders can be deduced as follows from that of expanders. Recalling (2.4), assume given a finite group G and $S \subset G$ as before such that $\varepsilon(G,S) = \varepsilon(\lambda_G, S) \geq \varepsilon > 0$. Recall that each $\sigma \in \hat{G}$ is contained in λ_G. Let $\pi \in \hat{G}$. Since any representation on G without invariant vectors, being a direct sum of non trivial irreps, is weakly contained in λ_G, the representation $\rho = \pi \otimes \overline{\pi}$ restricted to $H_\rho^{\mathrm{inv}\perp}$ is weakly contained in the non trivial part of λ_G. In particular, we have by (2.1)

$$\lambda(\rho, S) \leq \lambda(\lambda_G, S).$$

Therefore, we have

$$\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon(\lambda_G, S) \geq \varepsilon.$$

Thus if we are given an expander (G_k, S_k) as above, say with $S_k = \{s_1(k), \cdots, s_n(k)\}$, we can choose by (2.3) $\sigma_k \in \hat{G}_k$ such that $\dim(H_{\sigma_k}) \to \infty$, and if we set $u_j(k) = \sigma_k(s_j(k))$ $(1 \leq j \leq n)$, then $u(k) = \{u_1(k), \cdots, u_n(k)\}$ forms a quantum expander.

The next statement is a simple generalization of Proposition 2.1

Proposition 2.8. *For any $0 < \varepsilon < 1$ there is a constant $c'_\varepsilon > 0$ for which the following holds. Let G be any group and let $\pi : G \to B(H)$ be any unitary representation on a finite dimensional Hilbert space H. Let us assume that there is an n-element subset $S \subset G$ and $\varepsilon > 0$ such that*

$$\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon.$$

In other words, π satisfies the following spectral gap condition:

$$\lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon \tag{2.6}$$

Let $\pi = \oplus_{t \in T} \pi_t$ be the decomposition into distinct irreducibles (where each π_t has multiplicity $d_t \geq 1$), then

$$|\{t \in T \mid \dim(\pi_t) \leq N\}| \leq \exp c'_\varepsilon n N^2. \tag{2.7}$$

Proof. Let $\sigma = \oplus_{t \in T} \pi_t$ be the direct sum where each component is included with multiplicity equal to 1. We may clearly view σ as a subpresentation of π, acting on a subspace $K \subset H$ so that the orthogonal projection $Q : H \to K$ is intertwining, i.e. satisfies $Q\pi = \sigma Q$. Then we also have $(Q \otimes \bar{Q})(\pi \otimes \bar{\pi}) = (\sigma \otimes \bar{\sigma})(Q \otimes \bar{Q})$, from which it is easy to derive that if we denote $V_\pi = H^{\mathrm{inv}}_{\pi \otimes \bar{\pi}}$, we have $(Q \otimes \bar{Q})V_\pi = V_\sigma$ and $(Q \otimes \bar{Q})V_\pi^\perp = V_\sigma^\perp$. This implies

$$\lambda(\sigma \otimes \overline{\sigma}, S) \leq \lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon.$$

Thus, replacing π by σ, we may as well assume that the multiplicities d_t are all equal to 1.

Let $H = \oplus_{t \in T} H_t$ denote the decomposition corresponding to $\pi = \oplus_{t \in T} \pi_t$. We have $\pi \otimes \bar{\pi} = \oplus_{t,r \in T} \pi_t \otimes \overline{\pi_r}$, with associated decomposition $H \otimes \bar{H} = \oplus_{t,r \in T} H_t \otimes \overline{H_r}$. From this follows that the subspace $V_\pi \subset H \otimes \bar{H}$ of $\pi \otimes \bar{\pi}$-invariant vectors is equal to $\oplus_{t,r \in T} V_{t,r}$ where $V_{t,r} \subset H_t \otimes \overline{H_r}$ is the subspace of invariant vectors of $\pi_t \otimes \overline{\pi_r}$. Since for any $t \neq r \in T$, $\pi_t \not\simeq \pi_r$, by Schur's lemma $V_{t,r} = \{0\}$, and hence $V_\pi \subset \oplus_{t \in T} V_{t,t}$. In particular, this shows that

$$\forall t \neq r \in T \quad H_t \otimes \overline{H_r} \subset V_\pi^\perp.$$

Let $T' = \{t \in T \mid \dim(\pi_t) = N\}$. It suffices to show an estimate of the form

$$|T'| \le \exp c_\varepsilon n N^2. \tag{2.8}$$

Let $\mathcal{H}$ be the Hilbert space obtained by equipping M_N^n with the norm

$$\|x\|_{\mathcal{H}}^2 = N^{-1} n^{-1} \sum_1^n \operatorname{tr}(x_j^* x_j).$$

Let $S = \{s_1, \cdots, s_n\}$. For any $t \in T'$ we define $x(t) \in M_N^n$ by

$$x(t)_j = \pi_t(s_j) \quad 1 \le j \le n.$$

Note that, by our normalization, $\|x(t)\|_{\mathcal{H}} = 1$ for any $t \in T'$. Moreover, since for any $t \neq r \in T$ $\pi_t \not\simeq \pi_r$, by Schur's lemma the representation $\pi_t \otimes \overline{\pi_r}$ has no invariant vector, and hence lies inside $(\pi \otimes \overline{\pi})_{|V_\pi^\perp}$. Therefore, by (2.1)

$$\lambda(\pi_t \otimes \overline{\pi_r}, S) \le \lambda(\pi \otimes \overline{\pi}, S),$$

and hence for any unit vector $\xi \in H_{\pi_t} \otimes \overline{H_{\pi_r}}$ we have

$$n^{-1} \Re(\sum_{s \in S} \langle (\pi_t \otimes \overline{\pi_r})\xi, \xi \rangle) \le 1 - \varepsilon.$$

In particular, if $t \neq r \in T'$, we may realize π_t, π_r as representations on the same N-dimensional space, so that taking $\xi = N^{-1/2} I$ we find

$$\Re \langle x(t), x(r) \rangle_{\mathcal{H}} = (nN)^{-1} \Re \left(\sum_{s \in S} \operatorname{tr}(\pi_t(s)^* \pi_r(s)) \right) \le 1 - \varepsilon,$$

which implies

$$\|x(t) - x(r)\|_{\mathcal{H}} \ge \sqrt{2\varepsilon}.$$

Thus we have $|T'|$ points in the unit sphere of $\mathcal{H}$ that are $\sqrt{2\varepsilon}$-separated. Since $\dim(\mathcal{H}) = nN^2$, (2.8) follows immediately by a well known elementary volume argument (see e.g. [19, p. 57]). □

Remark 2.9. To derive Proposition 2.1 from the preceding statement, consider, in the situation of Proposition 2.1, a finite set $\{\sigma_t \mid t \in T\}$ of distinct finite dimensional irreducible representations of G, let π be their direct sum and let $\rho = \pi \otimes \overline{\pi}$. By the assumption in Proposition 2.1, we know $\varepsilon(\rho, S) \ge \varepsilon$, and hence (2.7) implies $|T| \le \exp c'_\varepsilon n N^2$. Applying this to $\pi = \lambda_G$, this shows that Proposition 2.8 contains Proposition 2.1.

For any finite dimensional unitary representation $\pi : G \to B(H)$ on an arbitrary group, let us denote by $r'_N(\pi)$ the number of distinct irreducible representations appearing in the decomposition of π of dimension at most N. Let then

$$R'_{n,\varepsilon}(N) = \sup r'_N(\pi)$$

where the sup runs over all π's and G's admitting an n-element generating set $S \subset G$ such that

$$\varepsilon(\pi \otimes \bar{\pi}, S) \ge \varepsilon.$$

Note that $r'_N(\lambda_G) = r_N(G)$ and hence

$$R_{n,\varepsilon}(N) \le R'_{n,\varepsilon}(N).$$

With this notation (2.7) means that

$$R'_{n,\varepsilon}(N) \le \exp c'_\varepsilon n N^2.$$

While it seems very difficult to give a good lower bound for $R_{n,\varepsilon}(N)$, we can answer the analogous question for $R'_{n,\varepsilon}(N)$: Indeed, the main result of [20] (see [20, Th. 1.3]), which follows, implies the desired lower bound when reformulated in terms of representations.

Theorem 2.10 ([20])**.** *For each $0 < \varepsilon < 1$, there is a constant $\beta_\varepsilon > 0$ such that and for all sufficiently large integer n (i.e. $n \ge n_0$ with n_0 depending on ε) and for all $N \ge 1$, there is a subset $\mathcal{T} \subset U(N)^n$ with*

$$|\mathcal{T}| \ge \exp \beta_\varepsilon n N^2$$

such that

$$\forall u \ne v \in \mathcal{T} \quad \|\sum\nolimits_1^n u_j \otimes \overline{v_j}\| \le n(1-\varepsilon) \quad \text{(we call these “}\varepsilon-\text{separated”)},$$

and $\varepsilon(u) \ge \varepsilon$ for all $u \in \mathcal{T}$ (we call these “ε-quantum expanders”).
More precisely, for all $u \in \mathcal{T}$ we have

$$\|(\sum u_j \otimes \overline{u_j})_{|I^\perp}\| \le n(1-\varepsilon).$$

Theorem 2.11. *The estimate in Proposition 2.8 is best possible in the sense that for any $0 < \varepsilon < 1$ there is a constant $\beta_\varepsilon > 0$ such that for any n large enough (i.e. $n \ge n_0(\varepsilon)$), for any $N \ge 1$ there is a group G and a finite dimensional representation π on G satisfying* (2.6) *and admitting a decomposition $\pi = \oplus_{t \in T} \pi_t$, with distinct irreducibles π_t each with multiplicity* 1 *(or any specified value ≥ 1) and acting on an N-dimensional space, with*

$$|T| \ge \exp \beta_\varepsilon n N^2.$$

Proof. Fix $N > 1$. Let $T \subset U(N)^n$ be the subset appearing in Theorem 2.10, i.e. T is such that $|T| \ge \exp \beta_\varepsilon n N^2$ and $\forall t \ne r \in T$ we have

$$\|\sum t_j \otimes \bar{r}_j\| \le n(1-\varepsilon), \tag{2.9}$$

and also

$$\|(\sum t_j \otimes \bar{t}_j)_{|I^\perp}\| \le n(1-\varepsilon). \tag{2.10}$$

Let $s_j = \oplus_{t \in T} t_j \in U(m)$ with $m = |T|N$, and let $G \subset U(m)$ be the subgroup generated by $S = \{s_1, \cdots, s_n\}$. Note that $\pi(G) \subset \oplus_{t \in T} M_N$. Let $\pi : G \to U(m)$ be the inclusion map viewed as a representation on G. Let $P_t : \oplus_{t \in T} M_N \to M_N$ be the $*$-homomorphism corresponding to the projection onto the coordinate of index t. For any $t \in T$, let $\pi_t : G \to U(N)$ be the representation defined by $\pi_t = P_t(\pi)$. Then, by definition, we have $\pi = \oplus_{t \in T} \pi_t$. By the spectral gap condition (2.10) the commutant of $\pi_t(S)$ (which is but the commutant of $\{t_1, \cdots, t_n\}$) is reduced to the scalars, so π_t is irreducible, and by (2.9) for any $t \ne r \in T$ the representations π_t and π_r are not unitarily equivalent. □

Remark 2.12. In particular, this means that $\forall n \ge n_0(\varepsilon)$ and $\forall N$

$$R'_{n,\varepsilon}(N) \ge \exp \beta_\varepsilon n N^2.$$

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References

[1] B. Bekka, P. de la Harpe and A. Valette, Kazhdan's property (T). Cambridge University Press, Cambridge, 2008.

[2] A. Ben-Aroya and A. Ta-Shma, Quantum expanders and the quantum entropy difference problem, 2007, arXiv:quant-ph/0702129.

[3] A. Ben-Aroya, O. Schwartz, and A. Ta-Shma, Quantum expanders: motivation and constructions. Theory Comput. 6 (2010), 47-79.

[4] J. Bourgain and A. Gamburd, Uniform expansion bounds for Cayley graphs of $SL_2(F_p)$, Ann. Math., 167 (2008), 625–642.

[5] M. Ershov and A. Jaikin-Zapirain, Property (T) for noncommutative universal lattices, Invent. Math. 179 (2010), no. 2, 303–347.

[6] P. de la Harpe, A.G. Robertson and A. Valette, On the spectrum of the sum of generators for a finitely generated group. Israel J. Math. 81 (1993), 6596.

[7] A. Harrow, Quantum expanders from any classical Cayley graph expander. Quantum Inf. Comput. 8 (2008), 715–721.

[8] M. Hastings and A. Harrow, Classical and quantum tensor product expanders. Quantum Inf. Comput. 9 (2009), 336–360.

[9] M. Hastings, Random unitaries give quantum expanders, Phys. Rev. A (3) 76 (2007), no. 3, 032315, 11 pp.

[10] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications. Bull. Amer. Math. Soc. 43 (2006), 439-561.

[11] M. Kassabov, Symmetric groups and expander graphs. Invent. Math. 170 (2007), no. 2, 327354.

[12] M. Kassabov, Universal lattices and unbounded rank expanders, Invent. Math. 170 (2007), no. 2, 297–326.

[13] M. Kassabov, A. Lubotzky and N. Nikolov, Finite simple groups as expanders. Proc. Natl. Acad. Sci. USA 103 (2006), 6116–6119.

[14] M. Kassabov and N. Nikolov, Cartesian products as profinite completions. Int. Math. Res. Not. 2006, Art. ID 72947, 17 pp.

[15] M. Larsen and A. Lubotzky, Representation growth of linear groups, J. Eur. Math. Soc. 10 (2008), 351–390.

[16] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures.* Progress in Math. 125. Birkhäuser, 1994.

[17] A. Lubotzky. Expander graphs in pure and applied mathematics, Bull. Amer. Math. Soc. 49 (2012) 113–162.

[18] R. Meshulam and A. Wigderson, Expanders in group algebras. Combinatorica 24 (2004), 659-680.

[19] G. Pisier, *The volume of Convex Bodies and Banach Space Geometry* . (Book) Cambridge University Press.1989.

[20] G. Pisier, Quantum Expanders and Geometry of Operator Spaces, J. Eur. Math. Soc. (JEMS) 16 (2014), 1183–1219.

[21] P. S. Wang, On isolated points in the dual spaces of locally compact groups. Math. Ann. 218 (1975), 19-34.

[22] S. Wassermann, C^*-algebras associated with groups with Kazhdan's property T. Ann. of Math. 134 (1991), 423-431.

[23] A. Wigderson, lecture notes for the 22nd mcgill invitational workshop on computational complexity, Bellairs Institute Holetown, Barbados Lecturers: Ben Green and AviWigderson.